THE FOUR POINTS

A pedagogical vignette

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1. Is it possible to locate four distinct points so that the distance between any pair of them is the same?

This is a good question to ask of elementary pupils. How can it be approached? If the question can be answered in the affirmative, then it is surely possible to find three such points, and the vertices of an equilateral triangle fill the bill uniquely. Now it is jus a matter of locating the fourth point.

Pupils may struggle aimlessly with this, until someone (perhaps the teacher) suggests that any three of the four points must form an equilateral triangle. Or, more transparently, the fourth point along with any two of the other three points must be the vertices of an equilateral triangle. Now it may be seen that the task is impossible.

Or is it? Most pupils will have an unconscious bias that leads to a negative answer, to wit that the four points lie in a plane. While many pupils will be convinced of the impossibility in the plane, an argument is not easy to come by and will present a challenge.

However, it may be that some child will realize that nothing in the statement of the problem restricts to two dimensions, and that if we are willing to entertain a third dimension, then the four vertices of a regular tetrahedron will serve. If no child considers this possibility, then it may be best to leave the problem in the plane, and see whether in the fulness of time, someone will think of solving the problem in space.

2. Determine all configurations of four distinct points in the plane for which each of the six distances between pairs of them is one of exactly two numbers.

At the outset, not all the pupils will understand what the problem is, and it may be necessary to measure the six pairwise ditances for four arbitrary points to illustrate the situation. If the pupils have been warmed up with Question 1, then some may readily discover that the four vertices of a 60-120-degree rhombus will serve; five of the six pairs of edges will have one length, the long diagonal will have another length.

However, I have found that fairly quickly, most classes will arrive at the four vertices of the square. This is a fortunate example, as it clarifies the problem for anyone who is uncertain. There is a more subtle issue that may emerge, the need to decide what the word *all* signifies. There are infinitely many instances of sets of four vertices of a square, and it does not generally take much convincing that we should agree that they are essentially the same and should be regarded as a single case. (The need to decide what is essentially the same and what is essentially different is an issue that frequently arises in mathematical classification problems.)

The hunt is on for other instances, and the enquiry might take one of several directions, depending on what ideas occur to the pupils. Some classes may wonder whether we can locate the four points collinearly. The argument that in this case, the pairwise distances involve at least three distinct distances is quite straightforward, but a challenge to children who are not used to mathematical argumentation. However, this is a good example to start on the road towards proof.

An approach that is regularly adopted by the groups I have tried this on is to start with the vertices of an equilateral triangle and then see how the fourth point can be appended. It turns out that there are four cases, and these can be discovered in any order by the children:

(a) the fourth point can be placed to obtain the 60-120-degree rhombus described above;
(b) the fourth point can be placed on the right bisector of one of the edges of the equilateral triangle on the same side as the third vertex so that its distance from that vertex is the sidelength of the triangle;
(c) the fourth point can be placed on the right bisector of one of the edges of the equilateral triangle on the opposite side as the third vertex so that its distance from that vertex is the sidelength of the triangle.
(d) the fourth point can be placed at the centroid of the other three;

To arrive at these cases some explicit or implicit analysis is required of the pupils. They will realize that in placing the fourth point, only one more value of the distance can be introduced, and this distance will be common to one, two or three pairs. If only one, then we are lead to case (a). If more than one, then the fourth point must lie on some right bisector. High school students may be familiar with this through their regular geometry class; elementary pupils may be able to intuit it. If the distance is common to two points, we are led to (b) and (c); if three, then to (d).

There is a sixth answer to the question, and often time will run out before it is found. This is not a bad thing, as it leave the class with something to think about and provides the opportunity for a later communication, possibly by email. This is the 72-108-degree equilateral trapezoid with three equal adjacent sides, the fourth side being equal to its two diagonals.

With an elementary class, usually the best you can hope for is for them to use their imaginations to envisage a possibility that does not involve an equilateral triangle. There is a mild application of the pigeonhole principles involved as they realize that one of the distances must be common to at least three pairs, and so the four points can be strung out in a broken line with the distances between adjacent pairs the same. How should that broken line be configured? (One way will lead us back to the rhombus.)

More advanced students can be assigned the task of determining what the ratio of the two distances and the angles of the trapezoid must be.

However, as with most challenges of this type, the leader can be pleasantly surprised by an insightful response and realize for the first time something about the situation. The occured for me on the occasion when the *very first* example of four points exemplifying two distances offered was the trapezoid. I told the pupil that this was the case that was generally missed, and wondered how he was able to find it so quickly. He simply said that you take four of the five vertices of a regular pentagon (the sides of which have one length and the diagonals another).

Although I have never reached this point with any group, one can also ask whether there are any further instances in space.

Concluding remarks. The problem has several advantages. It is an attractive challenge. Background requirements are virtually nil; pupils should be familiar with squares and equilateral triangles. There is an easy case that can almost be guaranteed to be found early in the session, and the remaining cases are more or less challenging to find. It lends itself to group work, as pupils can build on partial answers, and can approach the situation in a number of ways.

I have never tried this with geometric software, but it might be interesting to turn students loose on a computer to see what happens. It will surely change the solving dynamic. Using pencil and paper, there is a premium on coming up with some systematic analysis, whereas a computer lends itself more to continuous experimentation and the possibility of happening on solutions. It may be more difficult with a computer for a pupils to determine whether they have a complete set of solutions. Another issue that may arise with a computer is whether the structural aspects of the various possibilities remain hidden. For example, will a student realize the case (a) involves a rhombus?